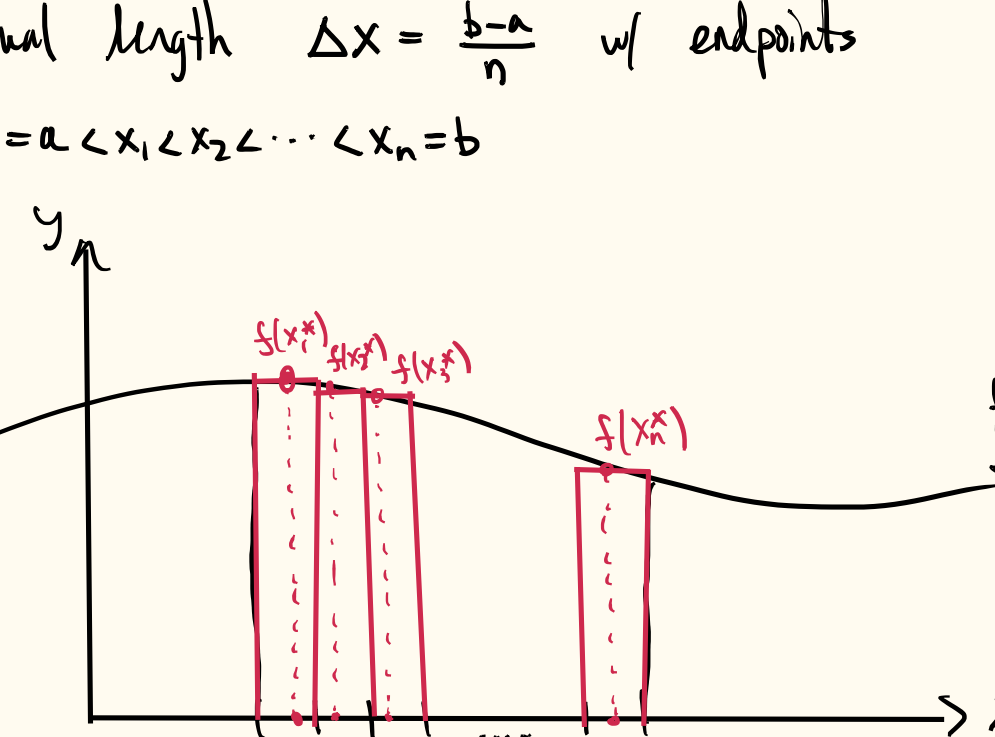


Section 11.1

Double Riemann Sums

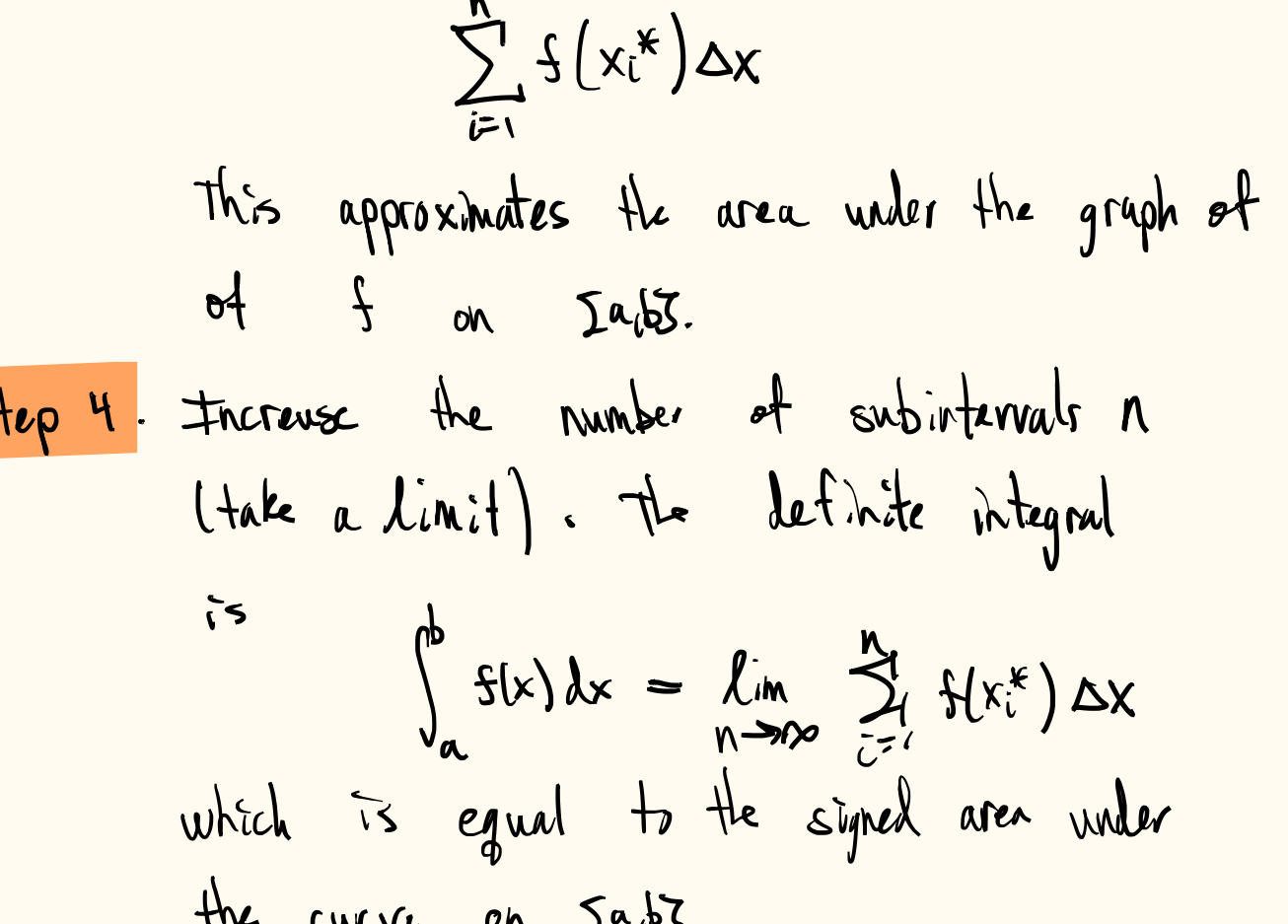
Double Integrals over Rectangles

Let $f(x)$ be a function that is continuous on $[a, b]$. The goal is to compute the (signed) area under the graph of f .



Step 1: Partition $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ w/ endpoints

$$x_0 = a < x_1 < x_2 < \dots < x_n = b$$



Step 2: Within each subinterval $[x_{i-1}, x_i]$ pick a "sample" point x_i^* . Form a rectangle w/ base $[x_{i-1}, x_i]$ and height $f(x_i^*)$.

Step 3: The area of each rectangle is $f(x_i^*)\Delta x$. Form the **Riemann Sum**

$$\sum_{i=1}^n f(x_i^*)\Delta x$$

This approximates the area under the graph of f on $[a, b]$.

Step 4: Increase the number of subintervals n (take a limit). The definite integral is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

which is equal to the signed area under the curve on $[a, b]$.

Section 11.1.2

Double Riemann Sums

Double Integrals

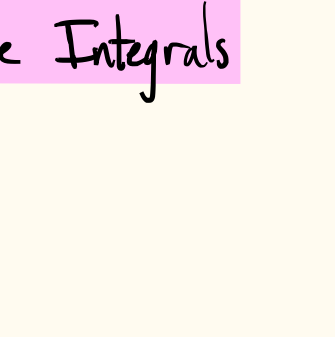
Let $f(x, y)$ be a function which is continuous on a rectangle $R = [a, b] \times [c, d]$.

A double Riemann sum is constructed as follows:

① Partition $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$ w/ endpoints $x_0 = a < x_1 < x_2 < \dots < x_m = b$.

② Partition $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$ w/ endpoints $y_0 = c < y_1 < \dots < y_n = d$.

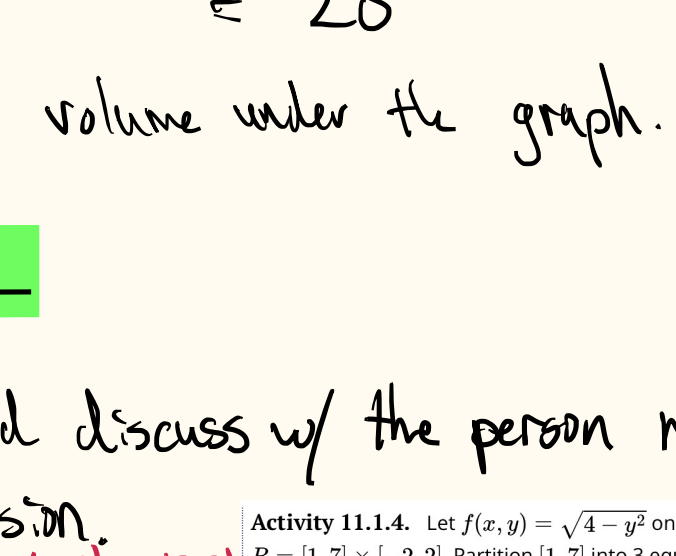
③ Together, ① and ② partition R into $m \cdot n$ rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ w/ areas $\Delta A = \Delta x \Delta y$



4. For each rectangle R_{ij} , choose a sample point (x_{ij}^*, y_{ij}^*) in R_{ij} . The double Riemann sum is

$$\sum_{j=1}^n \sum_{i=1}^m f(x_{ij}^*, y_{ij}^*) \Delta A$$

Each quantity $f(x_{ij}^*, y_{ij}^*) \Delta A$ is the signed volume of a rectangular prism



The double Riemann sum approximates the signed volume of the solid bounded by the graph of f and the xy -plane on R .

By increasing the number of subrectangles we can obtain the precise volume.

Definition Let $f(x, y)$ be a continuous function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. With the notation used above, the **double integral** of $f(x, y)$ over R is the limit

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{j=1}^n \sum_{i=1}^m f(x_{ij}^*, y_{ij}^*) \Delta A$$

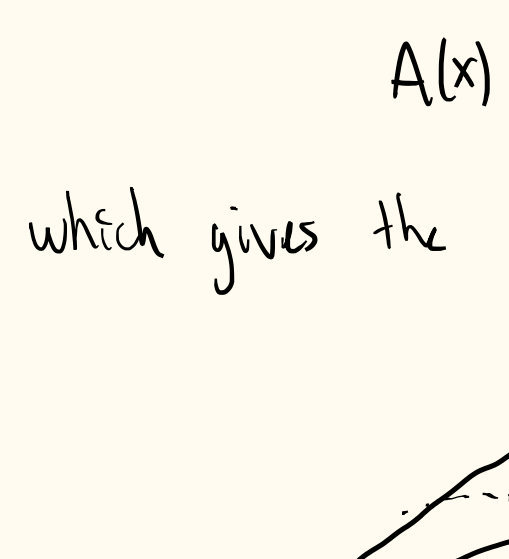
The double integral is equal to the signed volume bounded by the graph of $f(x, y)$ and the xy -plane over the region R .

Section 11.1.3

Interpretations of Double Integrals

Activity 11.1.3

- Complete and discuss w/ the person next to you.
- Class discussion.



$$(b) (-1.5, 1.5) \quad (-1.5, 2.5)$$

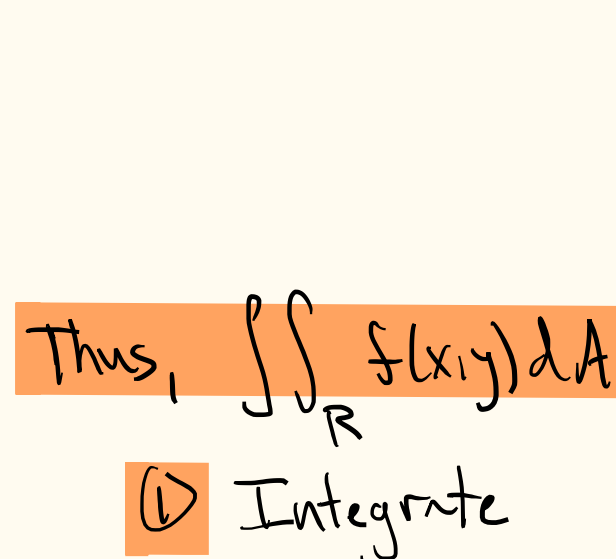
$$(1.5, 1.5) \quad (1.5, 2.5)$$

$$(c) \sum_{j=1}^2 \sum_{i=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A = f(-1.5, 1.5) + f(-1.5, 2.5) + f(1.5, 1.5) + f(1.5, 2.5) = 3 \cdot 5 + 5 \cdot 5 + 4 \cdot 5 + 6 \cdot 5 = 20$$

(d) Approximate volume under the graph.

Activity 11.1.4

- Complete and discuss w/ the person next to you.
- Class discussion.



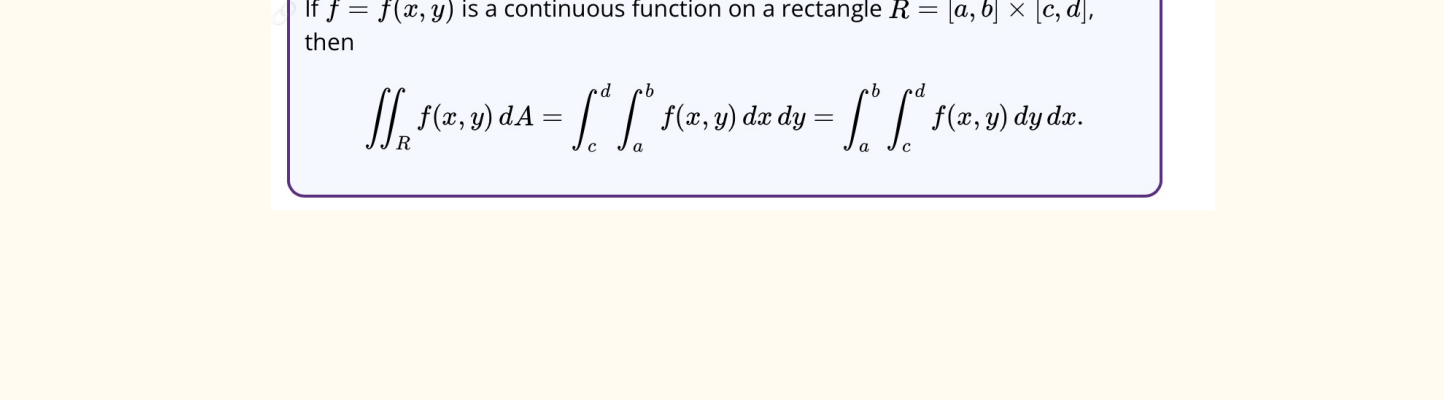
$$(b) \sum_{j=1}^3 \sum_{i=1}^3 f(x_{ij}^*, y_{ij}^*) \Delta A = (f(3, 0) + f(3, 1) + f(3, 2) + f(5, 0) + f(5, 1) + f(5, 2) + f(7, 0) + f(7, 1) + f(7, 2)) \cdot 1 = 24$$

$$(c) V = \frac{1}{2} \pi r^2 h = \frac{1}{2} \pi \cdot 4 \cdot 6 = 12\pi \approx 37.7$$

(d) Use more subrectangles.

Use better sample points.

Average two or more different approximations.



Section 11.2

Iterated Integrals

We can compute $\iint_R f(x, y) dA$ using the Fundamental Thm of Calculus.

Let $f(x, y)$ be continuous on a rectangle $R = [a, b] \times [c, d]$.

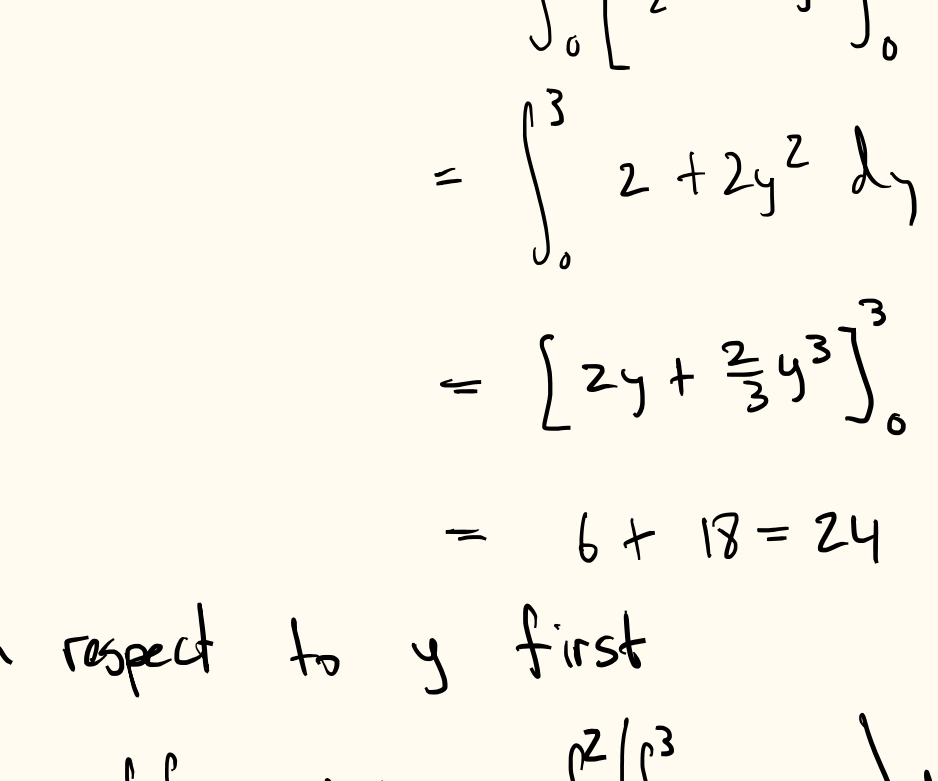
Fix $a \leq x_0 \leq b$. The area under the trace of $f(x, y)$ in the y -direction through x_0 is given by

$$\int_c^d f(x_0, y) dy$$

By letting x_0 vary, we obtain a function of one variable

$$A(x) = \int_c^d f(x, y) dy$$

which gives the area under a trace.



We can approximate the volume use $A(x)$ and a Riemann sum. Partition $[a, b]$ into n pieces as before. The red "cake slice" in the picture has volume

$$A(x_i^*) \Delta x$$

where x_i^* is a sample point. Thus, the double integral can be approximated as

$$\sum_{i=1}^n A(x_i^*) \Delta x$$

Taking a limit, we get

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x = \int_a^b A(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

Thus, $\iint_R f(x, y) dA$ can be computed as follows:

① Integrate $\int_c^d f(x, y) dy$ w/ respect to y while treating x as a constant.

② The result of ① is a function $A(x)$. Integrate $\int_a^b A(x) dx$ as usual.

The argument above can be performed using the trace in the x -direction instead to obtain Fubini's theorem:

(a) I want to integrate x first.

$$\iint_R f(x, y) dA = \int_0^3 \left(\int_0^2 (x + y^2) dx \right) dy = \int_0^3 \left[\frac{1}{2} x^2 + xy^2 \right]_0^2 dy = \int_0^3 (2 + 2y^2) dy = \left[2y + \frac{2}{3} y^3 \right]_0^3 = 6 + 18 = 24$$

(b) With respect to y first

$$\iint_R f(x, y) dA = \int_0^2 \left(\int_0^3 (x + y^2) dy \right) dx = \int_0^2 \left[xy + \frac{1}{3} y^3 \right]_0^3 dx = \int_0^2 (3x + 9) dx = \left[\frac{3}{2} x^2 + 9x \right]_0^2 = 6 + 18 = 24$$

Activity 11.2.3

- Complete and discuss w/ person next to you.
- Class discussion

(a) I want to integrate x first.

$$\iint_R f(x, y) dA = \int_0^3 \left(\int_0^2 (x + y^2) dx \right) dy = \int_0^3 \left[\frac{1}{2} x^2 + xy^2 \right]_0^2 dy = \int_0^3 (2 + 2y^2) dy = \left[2y + \frac{2}{3} y^3 \right]_0^3 = 6 + 18 = 24$$

(b) With respect to y first

$$\iint_R f(x, y) dA = \int_0^2 \left(\int_0^3 (x + y^2) dy \right) dx = \int_0^2 \left[xy + \frac{1}{3} y^3 \right]_0^3 dx = \int_0^2 (3x + 9) dx = \left[\frac{3}{2} x^2 + 9x \right]_0^2 = 6 + 18 = 24$$